International Journal of Theoretical Physics, Vol. 31, No. 12, 1992

Self-Dual Yang–Mills Fields in an Einstein Universe

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Received November 7, 1991

We find exact solutions of the self-consistent Einstein–Yang–Mills system of equations. These solutions are self-dual Yang–Mills fields in $I^1 \times S^3$ space-time.

1. INTRODUCTION

Instanton solutions of Yang–Mills (YM) fields in flat Euclidean space have physical importance in studying the gauge field vacuum (Belavin *et al.*, 1975). It is interesting to see how the YM fields are modified in a space-time with nontrivial topology. Chacrabarti (1987) found the YM instantons in Schwarzschild–de Sitter and other manifolds.

2. EXACT SOLUTIONS

Our goal is to study self-dual YM fields in the Einstein universe, where time is imaginary. The metric is

$$ds^2 = dt^2 + d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta \, d\varphi^2) \tag{1}$$

where t is the time and χ, θ, and φ are coordinates on the sphere S^3.

Let us consider a principal fiber bundle $P(I^1 \times S^3, SU(2))$ (Daniel and Viallet, 1980). The Yang–Mills fields (gauge potentials) A_μ are the coordinates of a connection form on the principal fiber bundle. The action functional for such fields is

$$S = -1/(16\pi) \int d^4x \, g^{1/2} g^{\mu\nu} g^{\alpha\beta} \langle F_{\mu\alpha}, F_{\nu\beta} \rangle \tag{2}$$

where $(g^{\mu\nu})$ is the inverse metric tensor, $g = \det(g_{\mu\nu})$, $\langle \cdot \cdot \cdot \rangle$ is the Killing

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0020-7748/92/1200-2061$06.50/0

form of the Lie algebra $su(2)$, and $F_{\mu\nu}$ are coordinates of the YM curvature form:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{3}$$

Self-dual fields are defined by the condition

$$F_{\mu\nu} = \tilde{F}_{\mu\nu} \equiv 1/2 g^{1/2} \varepsilon_{\mu\nu\alpha\beta} g^{\alpha\alpha'} g^{\beta\beta'} F_{\alpha'\beta'} \tag{4}$$

where $\varepsilon_{\mu\nu\alpha\beta}$ is the Levi-Civita tensor. Due to the Bianchi identity, the self-dual fields (4) identically satisfy the YM equations

$$g^{-1/2} \nabla_\mu (g^{1/2} F^{\mu\nu}) = 0 \tag{5}$$

where ∇_μ is the covariant derivative.

For self-dual fields in the Euclidean section the energy-momentum tensor is zero:

$$T_{\mu\nu} = 0 \tag{6}$$

Hence, these fields do not act on gravity, and it is sufficient in this case to solve the system of equations (4). According to the theory of symmetry with gauge compensation, one takes the ansatz (Burlankov, 1977)

$$\begin{aligned} A_t &= -(\Phi + n)\tau_3, \qquad A_\chi = 0 \\ A_\theta &= W\tau_2, \qquad A_\varphi = \cos\theta\,\tau_3 - W \sin\theta\,\tau_1 \end{aligned} \tag{7}$$

Φ, W are functions depending only on one variable χ; n is an integer, and the τ_k are expressed through the Pauli matrices:

$$\tau_k = \sigma_k/(2i) \tag{8}$$

The self-dual equations (4), expressed through the Lamé coefficients, have the form

$$\frac{F_{t\chi}}{h_t h_\chi} = \frac{F_{\theta\varphi}}{h_\theta h_\varphi}, \qquad \frac{F_{t\theta}}{h_t h_\theta} = -\frac{F_{\chi\varphi}}{h_\chi h_\varphi}, \qquad \frac{F_{t\varphi}}{h_t h_\varphi} = \frac{F_{\chi\theta}}{h_\chi h_\theta} \tag{9}$$

Using (3) and (7), we come to the system

$$\frac{d\Phi}{d\chi} = -\frac{1 - W^2}{\sin^2\chi}, \qquad \frac{dW}{d\chi} = (\Phi + n)W \tag{10}$$

Integrating (10) yields two families of analytic solutions ($\mu, m \in \mathbb{R}^1$):

$$\begin{cases} \Phi = -\mu \coth(\mu\chi) - n + \cot\chi \\ W = -\dfrac{\mu \sin\chi}{\sinh(\mu\chi)} \end{cases} \tag{11}$$

$$\begin{cases} \Phi = m \cot(m\chi) - n + \cot\chi \\ W = -\dfrac{m \sin\chi}{\sin(m\chi)} \end{cases} \tag{12}$$

To get the topological classification of these solutions, let us calculate the Chern index:

$$c_2 = 1/(16\pi^2) \int_0^\pi \mathrm{tr} \frac{F^{\mu\nu}}{h_\mu^2 h_\nu^2} \sin^2\chi \, d\chi \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\varphi \int_0^{T\to\infty} dt$$

$$= \int_0^\pi (\Phi'(1 - W^2) - 2W'(\Phi + n)W) \, d\chi$$

$$= (\Phi + n)(1 - W^2)|_0^\pi = \begin{cases} \infty & \text{for (11)} \\ 0 & \text{for (12)} \end{cases}$$

Consequently, our self-dual solutions have trivial topology and do not belong to instantons. It is possible that the result can help us to realize the vacuum structure of the fields under consideration.

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